

A formulation of a $(q + 1, 8)$ -cage

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Abstract

Let $q \geq 2$ be a prime power. In this note we present a formulation for obtaining the known $(q + 1, 8)$ -cages which has allowed us to construct small (k, g) -graphs for $k = q - 1, q$ and $g = 7, 8$. Furthermore, we also obtain smaller $(q, 8)$ -graphs for even prime power q .

Keywords: Cages, girth, Moore graphs, perfect dominating sets.

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1 Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow the book by Bondy and Murty [14] for terminology and notation.

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *girth* of a graph G is the number $g = g(G)$ of edges in a smallest cycle. For every $v \in V$, $N_G(v)$ denotes the *neighbourhood* of v , i.e. the set of all vertices adjacent to v . The *degree* of a vertex $v \in V$ is the cardinality of $N_G(v)$. Let $S \subset V(G)$, then we denote by $N_G(S) = \cup_{s \in S} N_G(s)$ and by $N_G[S] = S \cup N_G(S)$.

A graph is called *regular* if all the vertices have the same degree. A (k, g) -*graph* is a k -regular graph with girth g . Erdős and Sachs [15] proved the existence of (k, g) -graphs for all values of k and g provided that $k \geq 2$. Since then most work carried out has focused on constructing a smallest one (cf. e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16, 18, 21, 24, 25]). A (k, g) -*cage* is a k -regular graph with girth g having the smallest possible number of vertices. Cages have been intensely studied since they were introduced by Tutte [28] in 1947. More details about constructions of cages can be found in the recent survey by Exoo and Jajcay [17].

In this note we are interested in $(k, 8)$ -cages. Counting the number of vertices in the distance partition with respect to an edge yields the following lower bound on the order of a $(k, 8)$ -cage:

$$n_0(k, 8) = 2(1 + (k - 1) + (k - 1)^2 + (k - 1)^3). \quad (1)$$

A $(k, 8)$ -cage with $n_0(k, 8)$ vertices is called a Moore $(k, 8)$ -*graph* (cf. [14]). These graphs have been constructed as the incidence graphs of generalized quadrangles $Q(4, q)$ and $W(q)$ [12, 17, 27], which are known to exist for q a prime power and $k = q + 1$ and no example is known when $k - 1$ is not a prime power (cf. [11, 13, 19, 22]). Since they are incidence graphs, these cages are bipartite and have diameter 4.

In this note, we present in Definition 2.1 a formulation for obtaining the known $(q + 1, 8)$ -cages with $q \geq 2$ a prime power. Then we check in Theorem 2.1 that the graph Γ_q with such a labelling is a $(q + 1, 8)$ -cage, for each prime power $q \geq 2$. Finally, we describe in Section 3

the utility of this equivalent description for the known $(q + 1, 8)$ -cages, for constructing small $(q - 1, 8)$ -graphs [6] and $(q + 1, 7)$ -graphs [4, 5], that give rise to new and better upper bounds. Furthermore, we also obtain smaller $(q, 8)$ -graphs for even prime power q .

2 The formulation

We start by presenting the following graph:

Definition 2.1 *Let \mathbb{F}_q be a finite field with $q \geq 2$ a prime power and ϱ a symbol not belonging to \mathbb{F}_q . Let $\Gamma_q = \Gamma_q[W_0, W_1]$ be a bipartite graph with vertex sets $W_i = \mathbb{F}_q^3 \cup \{(\varrho, b, c)_i, (\varrho, \varrho, c)_i : b, c \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_i\}$, $i = 0, 1$, and edge set defined as follows:*

For all $a \in \mathbb{F}_q \cup \{\varrho\}$ and for all $b, c \in \mathbb{F}_q$:

$$N_{\Gamma_q}((a, b, c)_1) = \begin{cases} \{(w, aw + b, a^2w + 2ab + c)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, a, c)_0\} & \text{if } a \in \mathbb{F}_q; \\ \{(c, b, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, c)_0\} & \text{if } a = \varrho. \end{cases}$$

$$N_{\Gamma_q}((\varrho, \varrho, c)_1) = \{(\varrho, c, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_0\}$$

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Or equivalently

For all $i \in \mathbb{F}_q \cup \{\varrho\}$ and for all $j, k \in \mathbb{F}_q$:

$$N_{\Gamma_q}((i, j, k)_0) = \begin{cases} \{(w, j - wi, w^2i - 2wj + k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, j, i)_1\} & \text{if } i \in \mathbb{F}_q; \\ \{(j, w, k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, j)_1\} & \text{if } i = \varrho. \end{cases}$$

$$N_{\Gamma_q}((\varrho, \varrho, k)_0) = \{(\varrho, w, k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_1\};$$

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Note that ϱ is just a symbol not belonging to \mathbb{F}_q and no arithmetical operation will be performed with it.

Next, we will make use of the following induced subgraph B_q of Γ_q .

Notation 2.1 Let $B_q = B_q[V_0, V_1]$ be a bipartite graph with vertex set $V_i = \mathbb{F}_q^3$, $i = 0, 1$, and edge set $E(B_q)$ defined as follows:

$$\text{For all } a, b, c \in \mathbb{F}_q : N_{B_q}((a, b, c)_1) = \{(j, aj + b, a^2j + 2ab + c)_0 : j \in \mathbb{F}_q\}.$$

In order to proceed, we need the following definition of a q -regular bipartite graph H_q introduced by Lazebnik, Ustimenko and Woldar [21].

Definition 2.2 [21] Let \mathbb{F}_q be a finite field with $q \geq 2$. Let $H_q = H_q[U_0, U_1]$ be a bipartite graph with vertex set $U_r = \mathbb{F}_q^3$, $r = 0, 1$, and edge set $E(H_q)$ defined as follows:

$$\text{For all } a, b, c \in \mathbb{F}_q : N_{H_q}((a, b, c)_1) = \{(w, aw + b, a^2w + c)_0 : w \in \mathbb{F}_q\}.$$

Lazebnik, Ustimenko and Woldar proved that H_q given in Definition 2.2 is q -regular, bipartite, of girth 8 and order $2q^3$.

Lemma 2.1 The graph B_q is isomorphic to the graph H_q .

Proof Let H_q be the bipartite graph from Definition 2.2. Since the map $\sigma : B_q \rightarrow H_q$ defined by $\sigma((a, b, c)_1) = (a, b, 2ab + c)_1$ and $\sigma((x, y, z)_0) = (x, y, z)_0$ is an isomorphism, the result holds. ■

In what follows, we will obtain the graph Γ_q from the graph B_q adding some new vertices and edges. We need a preliminary lemma.

Lemma 2.2 Let B_q be the graph from Notation 2.1. For any given $a \in \mathbb{F}_q$, the vertices in the set $\{(a, b, c)_1 : b, c \in \mathbb{F}_q\}$ are mutually at distance at least four. And, for any given $i \in \mathbb{F}_q$, the vertices in the set $\{(i, j, k)_0 : j, k \in \mathbb{F}_q\}$ are mutually at distance at least four.

Proof Suppose that there exists a path of length two $(a, b, c)_1(w, j, k)_0(a', b', c')_1$ in B_q . By Definition 2.1, $j = aw + b = aw + b'$ and $k = a^2w + 2ab + c = a^2w + 2ab' + c'$. From both equation

we get $b = b'$ and $c = c'$ which implies that $(a, b, c)_1 = (a, b', c')_1$ contradicting that the path has length two. Similarly suppose that there exists a path of length two $(i, j, k)_0(a, b, c)_1(i, j', k')_0$. Reasoning similarly, we obtain $j = ai + b = j'$, and $c = a^2i - 2aj + k = a^2i - 2aj' + k'$ yielding $(i, j, k)_0 = (i, j', k')_0$ which is a contradiction. ■

Figure 1 shows a spanning tree of Γ_q with the vertices labelled according to Definition 2.1.

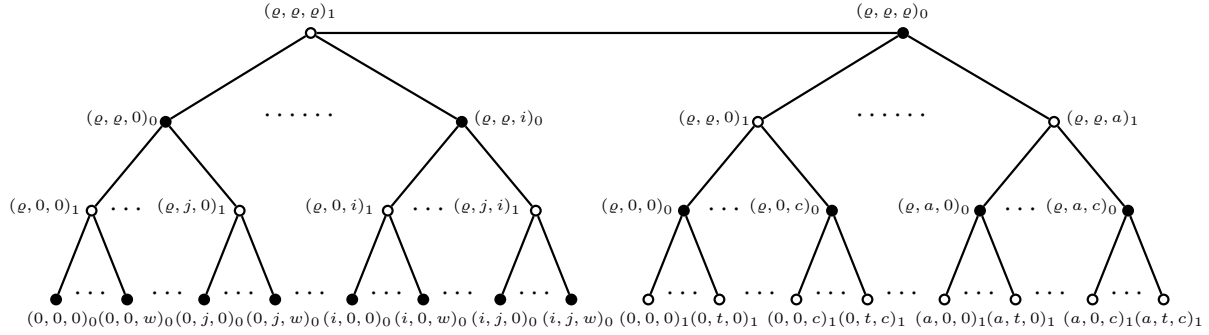


Figure 1: Spanning tree of Γ_q .

Theorem 2.1 *The graph Γ_q given in Definition 2.1 is a Moore $(q + 1, 8)$ -graph for each prime power $q \geq 2$.*

Proof As a consequence of Lemma 2.2, we obtain the following claim.

Claim 1: For all $x, y \in \mathbb{F}_q$, the q vertices of the set $\{(x, y, j)_0 : j \in \mathbb{F}_q\}$ are mutually at distance at least 6 in B_q .

Proof: By Lemma 2.2, the q vertices $\{(x, y, j)_0 : j \in \mathbb{F}_q\}$ are mutually at distance at least 4. Suppose by contradiction that B_q contains the following path of length four:

$$(x, y, j)_0 (a, b, c)_1 (x', y', j')_0 (a', b', c')_1 (x, y, j'')_0, \text{ for some } j'' \neq j.$$

Then $y = ax + b = a'x + b'$ and $y' = ax' + b = a'x' + b'$. It follows that $(a - a')(x - x') = 0$, which is a contradiction since by Lemma 2.2 $a \neq a'$ and $x \neq x'$. □

Let $B'_q = B'_q[V_0, V'_1]$ be the bipartite graph obtained from $B_q = B_q[V_0, V_1]$ by adding q^2 new vertices to V_1 labeled $(\varrho, b, c)_1$, $b, c \in \mathbb{F}_q$ (i.e., $V'_1 = V_1 \cup \{(\varrho, b, c)_1 : b, c \in \mathbb{F}_q\}$), and new edges $N_{B'_q}((\varrho, b, c)_1) = \{(c, b, j)_0 : j \in \mathbb{F}_q\}$ (see Figure 1). Then B'_q has $|V'_1| + |V_0| = 2q^3 + q^2$ vertices

such that every vertex of V_0 has degree $q + 1$ and every vertex of V_1' has still degree q . Note that the girth of B_q' is 8 by Claim 1. Further, Lemma 2.2 partially holds in B_q' . We write this fact in the following claim.

Claim 2: For any given $a \in \mathbb{F}_q \cup \{\varrho\}$, the vertices of the set $\{(a, b, c)_1 : b, c \in \mathbb{F}_q\}$ are mutually at distance at least four in B_q' .

Claim 3: For all $a \in \mathbb{F}_q \cup \{\varrho\}$ and for all $c \in \mathbb{F}_q$, the q vertices of the set $\{(a, t, c)_1 : t \in \mathbb{F}_q\}$ are mutually at distance at least 6 in B_q' .

Proof: By Claim 2, for all $a \in \mathbb{F}_q \cup \{\varrho\}$ the q vertices of $\{(a, t, c)_1 : t \in \mathbb{F}_q\}$ are mutually at distance at least 4 in B_q' . Suppose that there exists in B_q' the following path of length four:

$$(a, t, c)_1 (x, y, z)_0 (a', t', c')_1 (x', y', z')_0 (a, t'', c)_1, \text{ for some } t'' \neq t.$$

If $a = \varrho$, then $x = x' = c$, $y = t$, $y' = t''$ and $a' \neq \varrho$ by Claim 2. Then $y = a'x + t' = a'x' + t' = y'$ yielding that $t = t''$ which is a contradiction. Therefore $a \neq \varrho$. If $a' = \varrho$, then $x = x' = c'$ and $y = y' = t'$. Thus $y = ax + t = ax' + t'' = y'$ yielding that $t = t''$ which is a contradiction. Hence we may assume that $a' \neq \varrho$ and $a \neq a'$ by Claim 2. In this case we have:

$$\begin{aligned} y = ax + t &= a'x + t'; & z = a^2x + 2at + c &= a'^2x + 2a't' + c'; \\ y' = ax' + t'' &= a'x' + t'; & z' = a^2x' + 2at'' + c &= a'^2x' + 2a't' + c'. \end{aligned}$$

Hence

$$(a - a')(x - x') = t'' - t; \tag{2}$$

$$(a^2 - a'^2)(x - x') = 2a(t'' - t). \tag{3}$$

If q is even, (3) leads to $x = x'$ and (2) leads to $t'' = t$ which is a contradiction with our assumption. Thus assume q is odd. If $a + a' = 0$, then (3) gives $2a(t'' - t) = 0$, so that $a = 0$ yielding that $a' = 0$ (because $a + a' = 0$) which is again a contradiction. If $a + a' \neq 0$, multiplying equation (2) by $a + a'$ and subtracting both equations we obtain $(2a - (a + a'))(t'' - t) = 0$. Then $a = a'$ because $t'' \neq t$, which is a contradiction to Claim 2. Therefore, Claim 3 holds. \square

Let $B_q'' = B_q''[V_0', V_1']$ be the graph obtained from $B_q' = B_q'[V_0, V_1']$ by adding $q^2 + q$ new vertices to V_0 labeled $(\varrho, a, c)_0$, $a \in \mathbb{F}_q \cup \{\varrho\}$, $c \in \mathbb{F}_q$, and new edges $N_{B_q''}((\varrho, a, c)_0) = \{(a, t, c)_1 : t \in \mathbb{F}_q\}$ (see Figure 1). Then B_q'' has $|V_1'| + |V_0'| = 2q^3 + 2q^2 + q$ vertices such that every vertex has degree $q + 1$ except the new added vertices which have degree q . Moreover the girth of B_q'' is 8 by Claim 3.

Claim 4: For all $a \in \mathbb{F}_q \cup \{\varrho\}$, the q vertices of the set $\{(\varrho, a, j)_0 : j \in \mathbb{F}_q\}$ are mutually at distance at least 6 in B_q'' .

Proof: Clearly these q vertices are mutually at distance at least 4 in B_q'' . Suppose that there exists in B_q'' the following path of length four:

$$(\varrho, a, j)_0 (a, b, j)_1 (x, y, z)_0 (a, b', j')_1 (\varrho, a, j')_0, \text{ for some } j' \neq j.$$

If $a = \varrho$ then $x = j = j'$ which is a contradiction. Therefore $a \neq \varrho$. In this case $y = ax + b = ax + b'$ which implies that $b = b'$. Hence $z = a^2x + 2ab + j = a^2x + 2ab' + j'$ yielding that $j = j'$ which is again a contradiction. \square

Let $B_q''' = B_q'''[V_0'', V_1'']$ be the graph obtained from B_q'' by adding $q + 1$ new vertices to V_1' labeled $(\varrho, \varrho, a)_1$, $a \in \mathbb{F}_q \cup \{\varrho\}$, and new edges $N_{B_q'''}(\varrho, \varrho, a)_1 = \{(\varrho, a, c)_0 : c \in \mathbb{F}_q\}$, see Figure 1. Then B_q''' has $|V_1''| + |V_0'| = 2q^3 + 2q^2 + 2q + 1$ vertices such that every vertex has degree $q + 1$ except the new added vertices which have degree q . Moreover the girth of B_q''' is 8 by Claim 4 and clearly these $q + 1$ new vertices are mutually at distance 6. Finally, the Moore $(q + 1, 8)$ -graph Γ_q is obtained by adding to B_q''' another new vertex labeled $(\varrho, \varrho, \varrho)_0$ and edges $N_{\Gamma_q}((\varrho, \varrho, \varrho)_0) = \{(\varrho, \varrho, i)_1 : i \in \mathbb{F}_q \cup \{\varrho\}\}$. \blacksquare

Remark 2.1 A coordinatization of classical generalized quadrangles $Q(4, q)$ and $W(q)$ in four dimensions are discussed in [23, 26, 29]. The formulation of a Moore $(q + 1, 8)$ -graph given in Theorem 2.1 in three dimensions is equivalent to this coordinatization.

3 Applications

In this section we overview results that we have obtained for girth 7 in [4, 5] and for girth 8 in [6] using the labelling from Definition 2.1. Moreover, we also use the labelling to construct small $(q, 8)$ -graphs, for q even, which we match the bound on the order obtained by [18].

For girth 7, we have obtained the following results:

Theorem 3.1 ([4, Theorem 3.5], [5, Theorem 2.4]) *Let $q \geq 4$ be an even prime power. Then, there is a $(q + 1)$ -regular graph of girth 7 and order $2q^3 + q^2 + 2q$.*

Theorem 3.2 ([4, Theorem 4.7], [5, Theorem 3.4]) *Let $q \geq 5$ be an odd prime power. Then, there is a $(q + 1)$ -regular graph of girth 7 and order $2q^3 + 2q^2 - q + 1$.*

A subset $U \subset V(G)$ is said to be *perfect dominating set* of G if for each vertex $x \in V(G) \setminus U$, $|N_G(x) \cap U| = 1$ (cf. [20]). Note that if G is a k -regular graph and U is a perfect dominating set of G then $G - U$ is clearly a $(k - 1)$ -regular graph. In [6], using the labelling from Definition 2.1, we have obtained perfect dominating sets of $(q + 1, 8)$ -cages and $(q, 8)$ -graphs, for q a prime power, that give rise to the following two theorems:

Theorem 3.3 *Let $q \geq 4$ be a prime power. Then, there is a q -regular graph of girth 8 and order $2q(q^2 - 2)$.*

Theorem 3.4 *Let $q \geq 4$ be a prime power. Then, there is a $(q - 1)$ -regular graph of girth 8 and order $2q(q - 1)^2$.*

The labelling from Definition 2.1 also allows us to improve the result in Theorem 3.3 for even $q \geq 4$.

Proposition 3.1 *Let $q \geq 4$ be an even prime power and $\Gamma_q = \Gamma_q[V_0, V_1]$ the Moore $(q + 1, 8)$ -graph given in Definition 2.1. Let $Q = \{(\varrho, j, 0)_0 : j \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, 0)_0\}$ and $S = \{(u, u, 1 + u + u^2)_1 : u \in \mathbb{F}_q\} \cup \{(\varrho, 1, 1)_1\}$. Then the set*

$$N_{\Gamma_q}[Q] \cup \left(\bigcap_{a \in Q} N_{\Gamma_q}^2(a) \right) \cup N_{\Gamma_q}[S] \cup \left(\bigcap_{b \in S} N_{\Gamma_q}^2(b) \right)$$

is a perfect dominating set of Γ_q of cardinality $2(q^2 + 4q + 3)$ (see Figure 2).

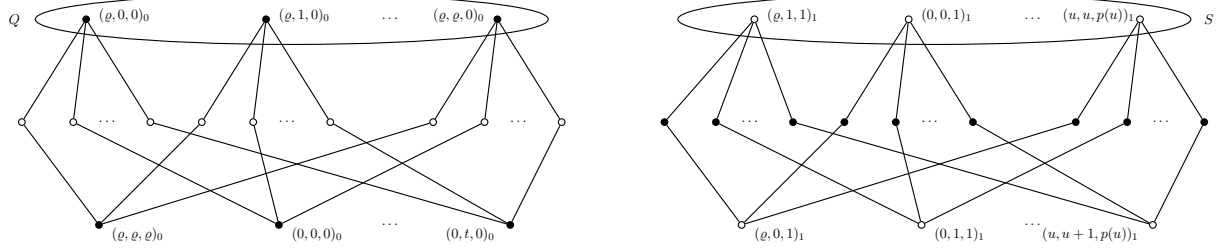


Figure 2: Perfect dominating set of Theorem 3.1.

Proof First assume that $q \geq 8$. By Definition 2.1, we have for all element of Q :

$$N_{\Gamma_q}((\varrho, j, 0)_0) = \{(j, t, 0)_1 : t \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, j)_1\};$$

$$N_{\Gamma_q}((\varrho, \varrho, 0)_0) = \{(\varrho, t, 0)_1 : t \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_1\}.$$

Then $(\varrho, \varrho, \varrho)_0 \in N_{\Gamma_q}^2((\varrho, j, 0)_0) \cap N_{\Gamma_q}^2((\varrho, \varrho, 0)_0)$ for all $j \in \mathbb{F}_q$. Moreover, since q is even, $2tj = 0$ and $N_{\Gamma_q}((j, t, 0)_1) = \{(w, jw + t, j^2w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, j, 0)_0\}$. Thus, for all $j_1, j_2 \in \mathbb{F}_q$, $j_1 \neq j_2$, we have $(w, j_1w + t_1, j_1^2w)_0 = (w, j_2w + t_2, j_2^2w)_0$ if and only if $w = 0$ and $t_1 = t_2$. Let $I_Q = \bigcap_{a \in Q} N_{\Gamma_q}^2(a)$. Then $I_Q = \{(0, t, 0)_0 : t \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_0\}$ (see left side of Figure 2) implying that $|N_{\Gamma_q}[Q]| + |I_Q| = (q + 1)^2 + 2(q + 1)$.

Let $p(u) = 1 + u + u^2$ and observe that $j = -j$ for all $j \in \mathbb{F}_q$ since q is even. By Definition 2.1, we have for all element of S :

$$N_{\Gamma_q}((u, u, p(u))_1) = \{(a, ua + u, u^2a + p(u))_0 : a \in \mathbb{F}_q\} \cup \{(\varrho, u, p(u))_0\};$$

$$N_{\Gamma_q}((\varrho, 1, 1)_1) = \{(1, 1, a)_0 : a \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, 1)_0\};$$

$$N_{\Gamma_q}((a, ua + u, u^2a + p(u))_0) = \{(w, ua + u + wa, w^2a + u^2a + p(u))_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, ua + u, a)_1\}.$$

$$N_{\Gamma_q}((\varrho, u, p(u))_0) = \{(u, y, p(u))_1 : y \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, u)_1\};$$

Then $(\varrho, 0, 1)_1 \in N_{\Gamma_q}((\varrho, \varrho, 1)_0) \cap N_{\Gamma_q}((1, 0, 1 + u)_0)$ for all $u \in \mathbb{F}_q$. Let $I_S = \bigcap_{b \in S} N_{\Gamma_q}^2(b)$, we have $(\varrho, 0, 1)_1 \in I_S$ (see right side of Figure 2). Moreover, two vertices $(a_1, a_1u_1 + u_1, a_1u_1^2 + p(u_1))_0$ and $(a_2, a_2u_2 + u_2, a_2u_2^2 + p(u_2))_0$ for $u_1, u_2, a_1, a_2 \in \mathbb{F}_q$ such that $u_1 \neq u_2$ and $a_1, a_2 \neq 1$ have a common neighbor $(w, x, y)_1$ for some $w, x, y \in \mathbb{F}_q$ if and only if

$$x = u_1a_1 + u_1 + wa_1 = u_2a_2 + u_2 + wa_2 \quad (4)$$

and

$$y = u_1^2 a_1 + w^2 a_1 + p(u_1) = u_2^2 a_2 + w^2 a_2 + p(u_2). \quad (5)$$

But (4) holds if and only if

$$x + w = (u_1 + w)(a_1 + 1) = (u_2 + w)(a_2 + 1). \quad (6)$$

Moreover, (5) holds if and only if

$$y + w^2 + 1 = (u_1^2 + w^2)(a_1 + 1) + u_1 = (u_2^2 + w^2)(a_2 + 1) + u_2. \quad (7)$$

Since q is even $(u_i^2 + w^2) = (u_i + w)^2$. Therefore multiplying (6) by $u_1 + w$ and combining it with (7) we get $u_1 = (u_2 + w)(a_2 + 1)(u_1 + u_2) + u_2$. But $u_1 \neq u_2$ gives that $(u_2 + w)(a_2 + 1) = 1$ as well as $(u_1 + w)(a_1 + 1) = 1$ by (6). Hence, $x = 1 + w$ by (6) and $y = 1 + w + w^2 = p(w)$ by (7). We conclude that $I_S = \{(u, 1 + u, p(u))_1 : u \in \mathbb{F}_q\} \cup \{(\varrho, 0, 1)_1\}$ since $(u, 1 + u, p(u))_1 \in N_{\Gamma_q}((\varrho, u, p(u))_0)$. Thus, $|N_{\Gamma_q}[S]| + |I_S| = (q + 1)^2 + 2(q + 1)$.

Let $D_Q = N_{\Gamma_q}[Q] \cup I_Q$ and $D_S = N_{\Gamma_q}[S] \cup I_S$. We get that $|D_Q \cup D_S| = 2(q + 1)^2 + 4(q + 1) = 2(q^2 + 4q + 3)$ since D_Q and D_S are vertex disjoint for $q \geq 8$.

Let us show that $D_Q \cup D_S$ is a perfect dominating set of Γ_q .

First, we check that by Definition 2.1, there is a matching joining each vertex of $D_Q \cap V_1$ with one vertex in $D_S \cap V_0$. We have:

- For all $u \in \mathbb{F}_q$, $(\varrho, \varrho, u)_1 \in D_Q$ is adjacent to $(\varrho, u, p(u))_0 \in D_S$, and $(\varrho, \varrho, \varrho)_1 \in D_Q$ is adjacent to $(\varrho, \varrho, 1)_0 \in D_S$.
- For all $t \in \mathbb{F}_q$, $(\varrho, t, 0)_1 \in D_Q$ is adjacent to $(0, t, p(t))_0 \in D_S$.
- For all $a \in \mathbb{F}_q$, $(a, a, 0)_1 \in D_Q$ is adjacent to $(1, 0, a^2)_0 \in N_{\Gamma_q}(\varrho, 0, 1)_1 \subset D_S$.
- For all $a \in \mathbb{F}_q$, $(a, a + 1, 0)_1 \in D_Q$ is adjacent to $(1, 1, a^2)_0 \in N_{\Gamma_q}(\varrho, 1, 1)_1 \subset D_S$.
- For all $a, t \in \mathbb{F}_q$, $t \neq 0, 1$, $(a, a + t, 0)_1 \in D_Q$ is adjacent to $(x, ax + a + t, a^2 x)_0 \in$

$N_{\Gamma_q}(u, u, p(u))_1 \subset D_S$ where $x, u \in \mathbb{F}_q$ are the solution of the system

$$\left. \begin{array}{lcl} a(x+1) + t & = & u(x+1) \\ a^2x & = & u^2(x+1) + u + 1 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{lcl} (x+1)(u+a) & = & t, \\ (a^2 + u^2)(x+1) + u + 1 & = & a^2. \end{array} \right.$$

Since q is even $(a^2 + u^2) = (a + u)^2$, yielding that $t(u + a) + u + 1 = a^2$, or equivalently $(t + 1)(u + a) = p(a)$. Consequently $u = a + (t + 1)^{-1}p(a)$ and $x = 1 + t(t + 1)p(a)^{-1}$.

Let H be the subgraph of Γ_q induced by $D_Q \cup D_S$. The existence of the matching joining each vertex of $D_Q \cap V_1$ with one vertex in $D_S \cap V_0$ allows us to conclude that the vertices of H have degree 3 or $q + 1$ in H , and the diameter of H is 5. Moreover since the girth is 8 we obtain $|N_{\Gamma_q}((D_Q \cup D_S)) \cap (V(\Gamma_q) \setminus (D_Q \cup D_S))| = 2(q - 2)(q + 1)^2 = 2(q^3 - 3q - 2) = |V(\Gamma_q) \setminus (D_Q \cup D_S)|$ yielding that $|N_{\Gamma_q}(v) \cap (D_Q \cup D_S)| = 1$ for all $v \in V(\Gamma_q) \setminus (D_Q \cup D_S)$. Therefore $D_Q \cup D_S$ is a perfect dominating set of Γ_q and the result holds for $q \geq 8$.

Finally, for $q = 4$, note that $p(x) = 1 + x + x^2 \in \{0, 1\}$ for all $x \in \mathbb{F}_4$ which implies that $N_{\Gamma_4}[S] \cup (\bigcap_{b \in S} N_{\Gamma_4}^2(b))$ and $N_{\Gamma_4}[Q] \cup (\bigcap_{b \in Q} N_{\Gamma_4}^2(b))$ defined in Proposition 3.1 are not vertex disjoint. Take $x \in \mathbb{F}_4 \setminus \{0, 1\}$, and let $Q' = \{(\varrho, j, x)_0 : j \in \mathbb{F}_4\} \cup \{(\varrho, \varrho, 0)_0\}$ and $S = \{(u, u, 1 + u + u^2)_1 : u \in \mathbb{F}_4\} \cup \{(\varrho, 1, 1)_1\}$. Then the set

$$N_{\Gamma_4}[Q'] \cup \left(\bigcap_{a \in Q'} N_{\Gamma_4}^2(a) \right) \cup N_{\Gamma_4}[S] \cup \left(\bigcap_{b \in S} N_{\Gamma_4}^2(b) \right)$$

is a perfect dominating set of the Moore $(5, 8)$ -graph of cardinality 70. Consequently by removing this set we obtain a 4-regular graph of girth 8 with 100 vertices. ■

Theorem 3.5 *Let $q \geq 4$ be an even prime power and $\Gamma_q = \Gamma_q[V_0, V_1]$ be the Moore $(q + 1, 8)$ -graph given in Definition 2.1. Then there is a q -regular graph of girth 8 and order $2(q^3 - 3q - 2)$.*

Proof The result is obtained by removing from Γ_q the perfect dominating set found in Proposition 3.1. ■

In [18] Gàcs and Héger have obtained $(q, 8)$ -bipartite graphs on $2q(q^2 - 2)$ vertices if q is odd, or on $2(q^3 - 3q - 2)$ vertices if q is even, using a regular point pair in a classical generalized

quadrangle. Note that in Theorem 3.5 we explicitly obtain $(q, 8)$ -bipartite graphs of the same cardinality using Definition 2.1. In [7] $(k, 8)$ -regular balanced bipartite graphs for all prime power q such that $3 \leq k \leq q$ of order $2k(q^2 - 1)$ have been obtained as subgraphs of the incidence graph of a generalized quadrangle. In [10] this result has been improved by constructing $(k, 8)$ -regular balanced bipartite graphs of order $2q(kq - 1)$.

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